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Asymptotics of wave motion in transversely isotropic plates

J.N. Sharma[a,*], Rajneesh Kumar[b]

[a] Department of Mathematics, National Institute of Technology, Deemed University, Hamirpur (H.P.) 177005, India
[b] Department of Mathematics, Kurukshetra University, Kurukshetra, Haryana 136119, India

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Abstract

The present investigation is concerned with elastic wave motion in infinite transversely isotropic plate by asymptotic method. The differential equations for the flexural and extensional motions are derived from the system of three-dimensional dynamical equations of linear elasticity. All coefficients of the differential operator are presented as explicit functions of the material parameter $\gamma = c_s^2/c_l^2$, the ratio of the squared velocities of flexural (shear) and extensional (longitudinal) waves. The velocity dispersion equations for the flexural and extensional wave motions are deduced analytically from the three-dimensional analog of Rayleigh–Lamb frequency equation for plates. The approximations for long and short waves are also obtained. The dispersion curves for phase velocity and group velocity spectrum are shown graphically for flexural and extensional wave motions of the plate. The results for isotropic materials have been deduced as special cases.

1. Introduction

In recent years considerable attention has been devoted to the problems of elastic wave motion in infinite plates because of its technical importance. Especially, notable are the works of Mindlin [1], Mindlin and Medick [2], Bache and Hegemier [3] and Losin [4,5]. Although a comprehensive review of the previous work is avoided here, we have to mention of some significant contributions to fields very closely related to the present topic, e.g., by Liu and Achenbach [6,7], Liu et al. [8–12], Liu and Tani [13], and Liu and Lam [14], on anisotropic linear solids, laminated plates, bars and strips. According to some recent publications [15–19] no simple direct relations between velocity, frequency and wave number for infinite plates are available at the present time. The asymptotic expansion of the frequency equation for flexural waves in a plate generated by the Rayleigh–Lamb

*Corresponding author. Tel.: +91-1972-223296; fax: +91-1972-223834.
E-mail address: jns@recham.ernet.in (J.N. Sharma).

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equation does not give an adequate approximation, even for velocities $v < c_S$. Moreover, three-dimensional wave propagation is possible at velocities $v < c_S$. All this motivates the search for some different approach to solution of the problem. The present study is an attempt to find a frequency and velocity dispersion relation from three-dimensional analog of the Rayleigh–Lamb frequency equation that would be adequate for flexural and extensional wave motions. The asymptotic method applied by Protsenko [20] for thin n-shelled structures is employed in this investigation.

2. Formulation of the problem

We consider free wave motion in a homogeneous transversely isotropic elastic plate of thickness $2h$, bounded by two stress free planes $z = \pm h$, and infinite in (x, y) directions. We formulate the corresponding dynamic boundary value problem of linear elasticity in matrix form:

$$\left\{ \begin{bmatrix} c_{44} & 0 & 0 \\ 0 & c_{44} & 0 \\ 0 & 0 & c_{33} \end{bmatrix} \frac{\partial^2}{\partial z^2} + \begin{bmatrix} 0 & 0 & (c_{13}+c_{44})\frac{\partial}{\partial x} \\ 0 & 0 & (c_{13}+c_{44})\frac{\partial}{\partial y} \\ (c_{13}+c_{44})\frac{\partial}{\partial x} & (c_{13}+c_{44})\frac{\partial}{\partial y} & 0 \end{bmatrix} \frac{\partial}{\partial z} \right.$$

$$\left. + \begin{bmatrix} c_{11}\frac{\partial^2}{\partial x^2} + \frac{1}{2}(c_{11}-c_{12})\frac{\partial^2}{\partial y^2} & \frac{1}{2}(c_{11}+c_{12})\frac{\partial^2}{\partial x \partial y} & 0 \\ \frac{1}{2}(c_{11}+c_{12})\frac{\partial^2}{\partial x \partial y} & \frac{1}{2}(c_{11}-c_{12})\frac{\partial^2}{\partial x^2} + c_{11}\frac{\partial^2}{\partial y^2} & 0 \\ 0 & 0 & c_{44}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \end{bmatrix} - \rho I \frac{\partial^2}{\partial t^2} \right\} \vec{U} = \vec{0}, \quad (1)$$

$$\vec{\tau} = \left\{ \begin{bmatrix} c_{44} & 0 & 0 \\ 0 & c_{44} & 0 \\ 0 & 0 & c_{33} \end{bmatrix} \frac{\partial}{\partial z} + \begin{bmatrix} 0 & 0 & c_{44}\frac{\partial}{\partial x} \\ 0 & 0 & c_{44}\frac{\partial}{\partial y} \\ c_{13}\frac{\partial}{\partial x} & c_{13}\frac{\partial}{\partial y} & 0 \end{bmatrix} \right\} \vec{U} = \vec{0}, \quad \text{on } z = \pm h, \quad (2)$$

where $\vec{U}(x, y, z, t) = (u, v, w)^{\mathrm{t}}$ is the displacement vector, $\vec{\tau} = (\tau_{Xz}, \tau_{Yz}, \tau_{Zz})^{\mathrm{t}}$ is the stress vector and I is the identity matrix. Here Eq. (1) is the system of equations of motion written in terms of elastic parameters c_{ij} and density ρ, the boundary conditions (2) express the absence of stresses at the free plate's surfaces $z = \pm h$.

We assume solutions in the form of a harmonic wave

$$\vec{U}(x, y, z, t) = \vec{u}(z) \exp\{-\mathrm{i}(k_x x + k_y y - \omega t)\}, \quad (3)$$

where $\vec{u}(z) = \{U(z), V(z), W(z)\}^{\mathrm{t}}$ is the amplitude vector, $\omega = \omega(\vec{k})$ is the circular frequency depending on the wave number $\vec{k} = (k_x, k_y)$.

Using Eq. (3) in Eqs. (1) and (2), we get

$$D\vec{u}'' - B_0(k)\vec{u}' - C_0(k)\vec{u} = 0, \quad -h < z < h, \quad (4)$$

$$\vec{\tau}(z, k) = D\vec{u}' - A_0(k)\vec{u} = 0, \quad z = \pm h, \quad (5)$$

where

$$B_0(k) = \mathrm{i}k\begin{bmatrix} 0 & 0 & (c_{13}+c_{44})n_1 \\ 0 & 0 & (c_{13}+c_{44})n_2 \\ (c_{13}+c_{44})n_1 & (c_{13}+c_{44})n_2 & 0 \end{bmatrix}, \qquad A_0(k) = \mathrm{i}k\begin{bmatrix} 0 & 0 & c_{44}n_1 \\ 0 & 0 & c_{44}n_2 \\ c_{13}n_1 & c_{13}n_2 & 0 \end{bmatrix},$$

$$C_0(k) = k^2\begin{bmatrix} c_{11}n_1^2 + \frac{1}{2}(c_{11}-c_{12})n_2^2 - \rho v^2 & \frac{1}{2}(c_{11}+c_{12})n_1n_2 & 0 \\ \frac{1}{2}(c_{11}+c_{12})n_1n_2 & \frac{1}{2}(c_{11}-c_{12})n_1^2 + c_{11}n_2^2 - \rho v^2 & 0 \\ 0 & 0 & c_{44}(n_1^2+n_2^2) - \rho v^2 \end{bmatrix},$$

$$D = \mathrm{diag}(c_{44}, c_{44}, c_{33}), \quad B_0(k) = A_0(k) + A_0^{\mathrm{t}}(k), \quad v(\vec{k}) = \omega(\vec{k})/k, \quad v(\vec{k}) = |\vec{v}(\vec{k})|,$$

$$\vec{u}' = \frac{\mathrm{d}\vec{u}(z)}{\mathrm{d}z}, \; k = |\vec{k}| = \sqrt{k_X^2 + k_Y^2}, \; \vec{k} = |\vec{k}|\vec{n} = k\vec{n}, \quad \vec{n} = (n_1, n_2), \; n_1 = k_X/k, \; n_2 = k_Y/k, \quad (6)$$

$\vec{v}$ is the phase velocity and v is the phase speed of a travelling wave and $\vec{n}$ is the unit direction vector. Eqs. (4) and (5) can finally lead to the boundary value problem in matrix form as the following system of ordinary differential equations:

$$\vec{u}'' = \hat{B}(k)\vec{u}' + \hat{C}(k)\vec{u}, \quad -h<z<h, \quad (7)$$

$$\tau_c(z,k) = \rho(\vec{u}' - \hat{A}(k)\vec{u}) = \vec{0}, \quad z = \pm h, \quad (8)$$

where

$$\hat{B}(k) = D^{-1}B_0(k) = kB, \quad \hat{C}(k) = D^{-1}C_0(k) = k^2C,$$

$$\hat{A}(k) = D^{-1}A_0(k) = kA, \quad D^{-1} = \mathrm{diag}(c_2^{-1}, c_2^{-1}, c_1^{-1}),$$

$$B = \mathrm{i}\begin{bmatrix} 0 & 0 & c_3n_1/c_2 \\ 0 & 0 & c_3n_2/c_2 \\ c_3n_1/c_1 & c_3n_2/c_1 & 0 \end{bmatrix}, \quad A = \mathrm{i}\begin{bmatrix} 0 & 0 & n_1 \\ 0 & 0 & n_2 \\ \frac{(c_3-c_2)n_1}{c_1} & \frac{(c_3-c_2)n_2}{c_1} & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} \frac{1}{c_2}n_1^2 + \frac{(1-c_4)}{c_2}n_2^2 - v_S^2 & \frac{c_4}{c_2}n_1n_2 & 0 \\ \frac{c_4}{c_2}n_1n_2 & \frac{(1-c_4)}{c_2}n_1^2 + \frac{1}{c_2}n_2^2 - v_S^2 & 0 \\ 0 & 0 & \frac{c_2}{c_1}(1-v_S^2) \end{bmatrix},$$

$$v_l^2 = \frac{v^2}{c_l^2}, \quad v_S^2 = \frac{v^2}{c_S^2}, \quad c_l^2 = \frac{c_{33}}{\rho}, \quad c_S^2 = \frac{c_{44}}{\rho},$$

$$c_1 = \frac{c_{33}}{c_{11}}, \quad c_2 = \frac{c_{44}}{c_{11}}, \quad c_3 = \frac{c_{13}+c_{44}}{c_{11}}, \quad c_4 = \frac{c_{11}+c_{12}}{2c_{11}},$$

$$\gamma = \frac{c_S^2}{c_l^2} = c_2/c_1, \quad c_s^2 = \gamma c_l^2, \quad v_l^2 = \gamma v_s^2.$$

Here $\tau_c(z,k) = \rho D^{-1}\tau(z,k) = (\tau_{xz}/c_s^2, \tau_{yz}/c_s^2, \tau_{zz}/c_l^2)$ is the modified stress vector, c_S, c_l are velocities (speeds) of the shear and extensional waves, respectively.

3. Asymptotic boundary value problem

We assume that $\tau_c(z,k)$ has a finite asymptotic expansion of the form

$$\tau_c(z,k) = \sum_{n=0}^{N} \tau_c^{(n)}(\xi)\varepsilon^n + o(\varepsilon^N), \quad \xi = \frac{z}{h}, \quad \varepsilon = kh,$$

and approximate it by Taylor series in $z(-h<z<h)$, about $z = 0$ so that

$$\tau_c(z) = \rho(\vec{u}' - \hat{A}(k)\vec{u}) = \sum_{n=0}^{N} \tau_c^{(n)}(0)z^n/n + o(z^N),$$

where the second argument is omitted for convenience. We denote $\tau_c^+ = \tau_c(h), \tau_c^- = \tau_c(-h)$ and combine them as

$$\tau_c^+ + \tau_c^- = 0, \quad \tau_c^+ - \tau_c^- = 0.$$

We obtain the boundary conditions in the asymptotic form:

$$\tau_c(0) + \frac{h^2}{2}\tau_c''(0) + \frac{h^4}{24}\tau_c^{(4)}(0) \approx 0, \tag{9}$$

$$\tau_c'(0) + \frac{h^2}{2}\tau_c'''(0) + \frac{h^4}{120}\tau_c^{(5)}(0) \approx 0. \tag{10}$$

Differentiating $\tau_c(z)$ with respect to z five times,

$$\tau_c^{(n)}(z) = \rho(\vec{u}^{(n+1)} - \hat{A}(k)\vec{u}^{(n)}), \quad n = 1,2,3,4,5, \tag{11}$$

and substituting Eq. (11) at $z = 0$ into Eqs. (9) and (10), we arrive at the asymptotic boundary value problem

$$\vec{u}'' = \hat{B}(k)\vec{u}' + \hat{C}(k)\vec{u}, \quad -h<z<h, \tag{12}$$

$$\vec{u}'(0) - \hat{A}(k)\vec{u}(0) + \frac{h^2}{2}(\vec{u}'''(0) - \hat{A}(k)\vec{u}''(0)) + \frac{h^4}{24}(\vec{u}^{(5)}(0) - \hat{A}(k)\vec{u}^{(4)}(0)) \approx 0, \tag{13}$$

$$\vec{u}''(0) - \hat{A}(k)\vec{u}'(0) + \frac{h^2}{6}(\vec{u}^{(4)}(0) - \hat{A}(k)\vec{u}'''(0)) + \frac{h^4}{120}(\vec{u}^{(6)}(0) - \hat{A}(k)\vec{u}^{(5)}(0)) \approx 0. \tag{14}$$

Eqs. (13) and (14) are valid at the free surfaces $z = \pm h$.

4. Resolving operator equation

The asymptotic boundary value problem, i.e., system of equations (12)–(14) can be reduced to one resolving operator equation written in terms of lambda matrices [21]. Upon differentiating

Eq. (12) four times we obtain

$$\vec{u}^{(n+2)} = \hat{B}(k)\vec{u}^{(n+1)} + \hat{C}(k)\vec{u}^{(n)}, \quad n = 1, 2, 3, 4,$$

and again using Eq. (12), we express $\vec{u}^n(0), n = 1, 2, 3, \ldots, 6$ in Eqs. (13) and (14) through $\vec{u}(0)$ and $\vec{u}'(0)$ with new matrix coefficients, as

$$\left(I + \frac{k^2h^2}{2}G + \frac{k^4h^4}{24}E\right)\vec{u}'(0) - k\left(A - \frac{k^2h^2}{2}H - \frac{k^4h^4}{24}F\right)\vec{u}(0) \approx 0, \tag{15}$$

$$\Rightarrow k\left(B - A + \frac{k^2h^2}{6}K + \frac{k^4h^4}{120}P\right)\vec{u}'(0) - k^2\left(C - \frac{k^2h^2}{6}L + \frac{k^4h^4}{120}Q\right)\vec{u}(0) \approx 0, \tag{16}$$

where

$$G = C + (B - A)B, \quad H = (B - A)C, \quad K = H + GB, \quad E = L + KB,$$

$$P = EB + F, \quad L = GC, \quad F = KC, \quad Q = EC.$$

Now, we consider the waves propagating along x-axis, so that $\vec{n} = (1, 0)$. In this case, the matrices G and E have the diagonal structures

$$G = \mathrm{diag}(g_{11}, g_{22}, g_{33}), E = \mathrm{diag}(e_{11}, e_{22}, e_{33}),$$

and the matrix coefficient

$$M = \mathrm{diag}(m_{11}, m_{22}, m_{33}),$$

in front of $\vec{u}'(0)$, in Eq. (15) has the form

$$M = I + \frac{k^2h^2}{2}G + \frac{k^4h^4}{24}E,$$

which leads to

$$m_{jj} = 1 + \frac{k^2h^2}{2}g_{jj} + \frac{k^4h^4}{24}e_{jj}, \quad j = 1, 2, 3. \tag{17}$$

Here

$$g_{11} = \frac{1}{c_2}\left[1 - c_2v_s^2 - \frac{(c_3 - c_2)c_3}{c_1}\right], \quad g_{22} = \frac{1}{c_2}[1 - c_4 - c_2v_s^2],$$

$$g_{33} = \gamma\left[1 - v_s^2 - \frac{c_3}{c_2}\right], \quad e_{11} = l_{11} - \frac{c_3}{c_1}k_{13}, \quad e_{22} = l_{22}, \quad e_{33} = l_{33} - \frac{c_3k_{31}}{c_2},$$

$$l_{11} = g_{11}\left(\frac{1}{c_2} - v_s^2\right), \quad l_{22} = g_{22}^2, \quad l_{33} = c_2(1 - v_s^2)g_{33}/c_1.$$

The diagonal matrix M is easily invertible,

$$M^{-1} = \mathrm{diag}\{m_{11}^{-1}, m_{22}^{-1}, m_{33}^{-1}\}, \ m_{jj} \neq 0,$$

and Eq. (15) can be solved for $\vec{u}'(0)$ as

$$\vec{u}'(0) \approx kM^{-1}\left(A - \frac{k^2h^2}{2}H - \frac{k^4h^4}{24}F\right)\vec{u}(0). \tag{18}$$

Substituting Eq. (18) into Eq. (16), we finally come to the resolving operator of the form

$$T\vec{u}(0) = \left(T_0 + \frac{k^2h^2}{6}T_2 + \frac{k^4h^4}{120}T_4\right)\vec{u}(0) \approx 0, \tag{19}$$

where

$$T_0 = C + (B - A)M^{-1}A, \quad T_2 = L + KM^{-1}A - 3(B - A)M^{-1}H,$$

$$T_4 = Q + PM^{-1}A - 10KM^{-1}H - 5(B - A)M^{-1}F.$$

The matrix of the operator T has a diagonal block structure (in general) where the blocks governing flexural (T_S) and in plane (T_L) motion are separated as

$$T\vec{u}(0) = \begin{bmatrix} T_L & 0 \\ 0 & T_S \end{bmatrix} \begin{bmatrix} u(0) \\ v(0) \\ w(0) \end{bmatrix} \approx 0, \tag{20}$$

where

$$T_L = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}, \quad T_S = t_{33}. \tag{21}$$

The system of Eq. (20) has non-trivial solution, if

$$\det T = \det T_L \det T_S = 0. \tag{22}$$

This implies that

$$\det T_L = 0, \tag{23}$$

$$\det T_S = 0 \quad \text{or} \quad t_{33} = 0. \tag{24}$$

Eq. (22) is the three-dimensional analog of the Rayleigh–Lamb frequency equation for a plate. Eqs. (23) and (24) are the corresponding frequency equation for extensional and flexural waves, respectively.

5. Flexural motion of a plate

Eq. (24) is the third equation of system (20) that governs the flexural vibrations, since the operator $T_S = t_{33}$ affects the displacement w only. This equation generates the dispersion equation and the velocity equation of a plate's flexural motion. According to structure (19) of the operator T, Eq. (24) has the form

$$t_{33}^{(0)} + \frac{k^2h^2}{6}t_{33}^{(2)} + \frac{k^4h^4}{120}t_{33}^{(4)} = 0, \tag{25}$$

where

$$t_{33}^{(0)} = -\frac{\gamma}{m_{11}}[1 - m_{11}(1 - v_S^2)],$$

$$t_{33}^{(2)} = l_{33} - \frac{k_{31}}{m_{11}} + 3\gamma(c_3 - c_2)(1 - v_S^2)/m_{11},$$

$$\begin{aligned} t_{33}^{(4)} = &\gamma(1 - v_S^2)e_{33} - \{c_3 e_{33}/c_1 + (1 - c_2 v_S^2)/c_2 k_{13}\}/m_{11} \\ &+ 10(c_3 - c_2)(1 - v_S^2)k_{31}/c_1 m_{11} + 5\gamma^2(1 - v_S^2)k_{13}/m_{11}. \end{aligned} \quad (26)$$

Eq. (25) is the velocity equation for the flexural vibration of a plate. For isotropic materials, we have

$$c_{11} = c_{33} = \lambda + 2\mu, \quad c_{44} = \mu, \quad c_{13} = c_{12} = \lambda, \text{ so that}$$

$$c_1 = 1, \quad c_2/c_1 = \gamma = \frac{\mu}{\lambda + 2\mu}, \quad c_3 = 1 - \gamma = c_4. \quad (27)$$

As a consequence, in this case, we have

$$t_{33}^{(0)} = \frac{-\gamma}{m_{11}}[1 - m_{11}(1 - v_S^2)], \quad t_{33}^{(2)} = \frac{1}{m_{11}}(m_{11}L_{33} - K_{31} + 3\gamma H_{13}),$$

$$t_{33}^{(4)} = \frac{\gamma}{m_{11}}\left[m_{11}(1 - v_S^2)E_{33}^* + \left(1 - \frac{1}{\gamma}\right)E_{33}^* - \frac{F_{31}^*}{\gamma} + \frac{10H_{31}k_{31}}{\gamma} + \frac{5}{\gamma}F_{13}^*\right],$$

where

$$L_{33} = \gamma^2\left[\left(2 - \frac{1}{\gamma}\right) - v_S^2\right](1 - v_S^2), \quad K_{31} = \gamma(3 - 2\gamma) - (2 - \gamma)v_S^2,$$

$$H_{13} = (1 - 2\gamma)(1 - v_S^2), \quad H_{31} = 1 - v_S^2, \quad E_{33}^* = E_{33}v_S^4 + E_{31}v_S^2 - E_{30},$$

$$F_{31}^* = F_{10} - F_{11}v_S^2 + F_{12}v_S^4, \quad F_{13}^* = F_{30} - F_{31}v_S^2 + F_{32}v_S^4. \quad (28)$$

The quantities E_{ij}, F_{ij} in Eq. (28) can be computed from the relevant relations of the previous section by making use of relation (17). The value of $t_{33}^{(0)}$ has been wrongly calculated in the case of isotropic material by Losin [4] and consequently some of the expressions and corresponding results as well as conclusions drawn by him following Eqs. (27) of his paper are incorrect. Substitution of Eq. (26) into Eq. (25) gives us

$$\begin{aligned} &b_0 v_S^{10} - \left(b_1 + \frac{20}{k^2h^2}a_0\right)v_S^8 + \left(b_2 + \frac{20}{k^2h^2}a_1 + \frac{120}{k^4h^4}C_0\right)v_S^6 \\ &- \left(b_3 + \frac{20}{k^2h^2}a_2 + \frac{120}{k^2h^2}C_1\right)v_s^4 + \left(b_4 + \frac{20}{k^2h^2}a_3 + \frac{120}{k^2h^2}C_2\right)v_S^2 \\ &- \left(b_5 + \frac{20}{k^2h^2}a_4 + \frac{120}{k^4h^4}C_3\right) = 0, \end{aligned} \quad (29)$$

where

$$b_0 = E_{33}, \quad a_0 = \frac{3}{5}E_{33} + \gamma, \quad C_0 = 1 - \frac{E_{33}}{5} - 2\gamma,$$

$$b_1 = E_{33} - E_{31} - E_{33}E_{11}, \quad a_1 = \frac{3}{5}(E_{33}G_{10} + E_{33} - E_{31}) + \gamma E_{11},$$

$$C_1 = 1 - E_{11} - 2(2\gamma - c_3/c_1) + 2\gamma G_{10} + \frac{1}{5}(E_{33} - E_{31}),$$

$$a_2 = \gamma E_{10} + \left(2\gamma - \frac{c_3}{c_1}\right)E_{11} - \left(\frac{c_3}{c_1} - \gamma\right) + \frac{3}{5}(E_{13} + E_{30}) - \frac{3}{5}(E_{33} + E_{31}),$$

$$b_2 = E_{10}E_{33} + E_{11}(E_{31} - E_{33}) - E_{31} - E_{30},$$

$$C_2 = 2(2\gamma - c_3/c_1)G_{10} - 2(c_3/c_1 - \gamma) - \frac{1}{5}(E_{31} + E_{30}),$$

$$a_3 = \frac{3}{5}(1 + G_{10} - E_{30}) + (2\gamma - c_3/c_1)E_{10} + (c_3/c_1 - \gamma)E_{11} - \frac{3}{5}G_{10}(E_{31} + E_{30}) - 10\frac{(c_3 - c_2)}{c_2}(c_1 + c_3) - 5F_{12},$$

$$C_3 = E_{10} - 2G_{10}(c_3/c_1 - \gamma) - \frac{E_{30}}{5}, \quad a_4 = -\frac{3}{5}E_{30}G_{10} - (c_3/c_1 - \gamma)E_{10},$$

$$b_4 = E_{11}E_{30} + \frac{c_3}{c_2}E_{33} + \gamma F_{32} - 10\left(\frac{c_3 - c_2}{c_1 c_2}\right)\left[(c_1 + c_3)c_2 + 1 - \frac{c_3(c_3 - c_2)}{c_1}\right] - 5F_{11} - E_{11} - E_{10} - E_{10}(E_{31} + E_{30}),$$

$$b_5 = \gamma F_{30} - \frac{c_3}{c_2}E_{30} + 10\frac{(c_3 - c_2)}{c_1^2 c_2}[c_1^2 - c_3(c_3 - c_2)] + 5F_{10} - E_{30}E_{10},$$

$$E_{10} = \frac{c_1 - c_3(c_3 - c_2)}{c_1^2 c_2^2} - \frac{c_1 c_3^2 - c_3(c_2 + c_3)(c_3 - c_2)^2}{c_1^2 c_2^2}, \tag{30}$$

$$E_{11} = \frac{c_3^2(c_1 + c_2) - c_3 c_2^2}{c_1^2 c_2} - \frac{2c_1 - c_3(c_3 - c_2)}{c_1 c_2}, \quad E_{30} = \frac{c_2^2(c_3 - c_2) - c_1 c_3 - c_3^2(c_3 - c_2)}{c_1^2 c_2},$$

$$E_{31} = \frac{c_2 - c_3(c_1 + c_3)}{c_1 c_2} - \frac{c_2^2}{c_1^2} + \frac{c_2(c_3 - c_2)}{c_1^2}, \quad E_{33} = \gamma^2,$$

$$F_{10} = \frac{c_1 c_3 - (c_2 + c_3)(c_3 - c_2)^2}{c_1^2 c_2}, \quad F_{12} = \frac{c_3(c_2 - c_1) - c_2^2}{c_1^2}, \quad F_{30} = \frac{c_1 - 2c_3 + c_2}{c_1^2 c_2},$$

$$F_{31} = c_2 F_{30} + \frac{1}{c_2}F_{32}, \quad F_{32} = \frac{c_2(c_1 + c_3)}{c_1}, \quad G_{10} = \frac{1}{c_2} - \frac{c_3(c_3 - c_2)}{c_1 c_2}. \tag{31}$$

6. Long and short wave approximations

For long waves, the wavelength is very large compared to the thickness $2h$ of the plate, i.e., $kh \to 0$. The limiting form of Eq. (29) after taking $v_s = \omega_s/k$ is given by

$$\omega_s^6\left[\omega_s^4 - \frac{20}{h^2}\left(\frac{3}{5} + \gamma\right)\omega_s^2 + \left(\gamma^2 - \frac{1}{5} + 2\gamma\right)\right] = 0. \tag{32}$$

This equation obviously has three trivial roots $\omega_s = 0$ and the corresponding phase velocity is also equal to zero. The quadratic equation in Eq. (32) with coefficients depending on h and γ, gives two more roots in the general case.

For short wave approximation, the wavelength is very small compared to the thickness of a plate. The substitution of $kh \to \infty$ in Eq. (29) gives the limiting form of the velocity equation:

$$b_0 v_s^{10} - b_1 v_s^8 + b_2 v_s^6 - b_3 v_s^4 + b_4 v_5^2 - b_5 = 0. \tag{33}$$

The roots of this equation are phase velocities of short wave modes depending on the material parameter γ only. This equation has five finite real roots that correspond to the velocities of the first five wave modes. The fundamental mode approaches the velocity of Rayleigh's surface wave c_R.

7. Extensional motion of a plate

The velocity equation for extensional motion of a plate is given by Eq. (23), viz.,

$$\det T_L = 0 \quad \text{or} \quad t_{11} t_{22} = 0. \tag{34}$$

The diagonal structure of the operator T implies that the system of propagation equations has the form $t_{jj} u_j = 0$ ($j = 1, 2, 3$) where each of the equation affects only one of the displacement components $u_1 = u, u_2 = v, u_3 = w$ and hence all the three equations being independent can be analyzed separately. Therefore, we can take $t_{11} u = 0$ as a propagation equation that governs the longitudinal wave motion along x-axis and hence equation $t_{11} u = 0$ that follows from Eq. (34) is a corresponding dispersion relation, which generates the frequency and velocity of the plate's extensional motion and has the form

$$\begin{aligned} &R_0 v_s^{10} + \left(R_1 - \frac{20 R_0}{k^2 h^2}\right) v_s^8 + \left(R_2 + \frac{20}{k^2 h^2} s_1 + \frac{120}{k^4 h^4} R_0\right) v_s^6 \\ &\quad + \left(R_3 + \frac{20}{k^2 h^2} s_2 + \frac{120}{k^4 h^4} P_1\right) v_s^4 + \left(R_4 + \frac{20}{k^2 h^2} s_3 + \frac{120}{k^4 h^4} P_2\right) v_s^2 \\ &\quad + \left(R_5 + \frac{20}{k^2 h^2} s_4 + \frac{120}{k^4 h^4} P_3\right) = 0, \end{aligned} \tag{35}$$

where

$$R_0 = -E_{33}, \quad R_1 = \frac{E_{33}}{c_2} - M_{33} - E_{11} E_{33}, \quad R_2 = \frac{1}{c_2}(M_{33} + E_{11} E_{33}) - (M_{33} E_{11} + M_{31}),$$

$$R_3 = \frac{1}{c_2}(E_{11} M_{33} + M_{31}) - (E_{33} M_{33} + E_{11} M_{31}), \quad R_4 = \frac{1}{c_2}(E_{33} M_{33} + E_{11} M_{31}) - E_{33} M_{31},$$

$$R_5 = \frac{1}{c_2} E_{33} M_{31}, \quad S_0 = -R_0, \quad S_1 = M_{33} - \left[\frac{1}{c_2} + \frac{c_1 - c_3(c_3 - c_2)}{c_1 c_2} + \frac{3(c_3 - c_2)}{c_1}\right] E_{33},$$

$$S_2 = M_{31} - M_{33}\left\{\frac{1}{c_2} + \frac{c_1 - c_3(c_3 - c_2)}{c_1 c_2} + \frac{3(c_3 - c_2)}{c_1}\right\} + E_{33}\left[\frac{3(c_3 - c_2)}{c_1} + \frac{c_1 - c_3(c_3 - c_2)}{c_1 c_2^2}\right],$$

$$S_3 = M_{33}\left\{\frac{3(c_3 - c_2)}{c_1 c_2} + \frac{c_1 - c_3(c_3 - c_2)}{c_1 c_2^2}\right\} - M_{31} - \left[\frac{1}{c_2} + \frac{c_1 - c_3(c_3 - c_2)}{c_1 c_2} + \frac{3(c_3 - c_2)}{c_1}\right],$$

$$S_4 = M_{31} - \left\{\frac{c_1 - c_3(c_3 - c_2)}{c_1 c_2^2} + \frac{3(c_3 - c_2)}{c_1 c_2}\right\}, \quad P_0 = R_0,$$

$$P_1 = \left(\frac{E_{33}}{c_2} - M_{33}\right) + \frac{Q_1}{5}, \quad P_2 = \left(\frac{M_{33}}{c_2} - M_{31}\right) + \frac{Q_1}{5}, \quad P_3 = \left(\frac{M_{31}}{c_2} + \frac{Q_3}{5}\right).$$

Eq. (35), in case of short wave ($kh \to \infty$) reduces to

$$R_0 v_s^{10} + R_1 v_s^8 + R_2 v_s^6 + R_3 v_3^4 + R_4 v_2^2 + R_5 = 0, \tag{36}$$

and in case of long waves ($kh \to 0$) Eq. (35) takes the form

$$\omega_s^6\left[R_0\omega_s^4 + \frac{20 s_0}{h^2}\omega_s^2 + \frac{120 R_0}{h^4}\right] = 0. \tag{37}$$

The second equation of the diagonalized system, $t_{22}v = 0$, governs the inplane wave motion in the same direction and affects the v component of the displacement vector. Its dispersion relation is $t_{22} = 0$, which provides us $v^2 = c_s^2$ as a unique real root of multiplicity one for long wave and of multiplicity three for short wave asymptotics which agrees with Losin [5] (cf. Eq. (19) in which the factors h^2 and h^4 are missing in the second and third terms under the braces).

8. Numerical results and discussions

With the view to illustrate and verify the theoretical developments in the previous sections, we present some numerical results in order to explain some hidden basic features of the extensional and flexural mode of wave propagation in an infinite homogenous transversely isotropic plate. The material chosen for this purpose is magnesium, the physical data of which is given by Sharma and Singh [22].

$$c_{11} = 5.975 \times 10^{10}\ \mathrm{N\,m^{-2}}, \quad c_{12} = 2.625 \times 10^{10}\ \mathrm{N\,m^{-2}}, \quad c_{13} = 2.17 \times 10^{10}\ \mathrm{N\,m^{-2}},$$

$$c_{33} = 6.17 \times 10^{10}\ \mathrm{N\,m^{-2}}, \quad c_{44} = 1.639 \times 10^{10}\ \mathrm{N\,m^{-2}}, \quad f = 1.74 \times 10^{3}\ \mathrm{kg\,m^{-3}}.$$

The dispersion curves for phase and group velocity for the first (fundamental) and second modes of extensional and flexural waves computed from relations (25) and (34) are given in Figs. 1 and 2 respectively. All the modes are found to be dispersive in character. The iteration method has been used to solve these dispersion equations. This method requires to put the equation $f(v) = 0$ into the form $v = g(v)$, so that the sequence $\{v_n\}$ of iteration for the desired root can be easily generated as follows: If v_0 is the initial approximation to the root, then $v_1 = g(v_0), v_2 = g(v_1), v_3 = g(v_2)$, etc. In general, $v_{n+1} = g(v_n), n = 0, 1, 2, 3, \ldots$. If $|g'(v)| \ll 1$, for all $v \in I$, then the sequence $\{v_n\}$ of approximations to the root will converge to the actual value $v = \zeta$ of the root, provided $v_0 \in I$ where I is the interval where the root is expected. This process is repeated time and again for a particular value of the non-dimensional wave number kh, unless the sequence of iteration to the value of v converges to desired level of accuracy, i.e., $|v_{n+1} - v_n| \angle \varepsilon$, ε being an arbitrary small number to be selected at random in order to achieve the accuracy level. The procedure has been

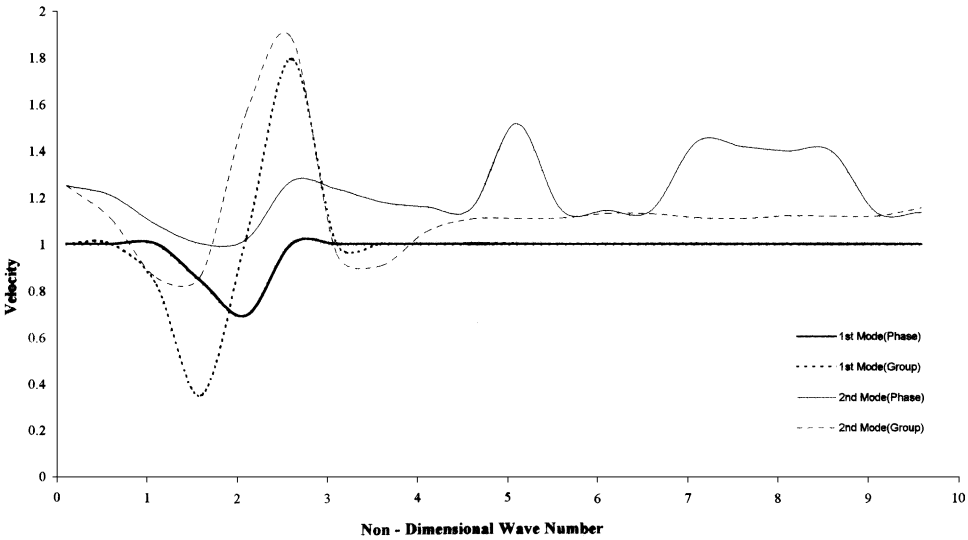

Fig. 1. Variation of phase and group velocities of extensional modes with wave number.

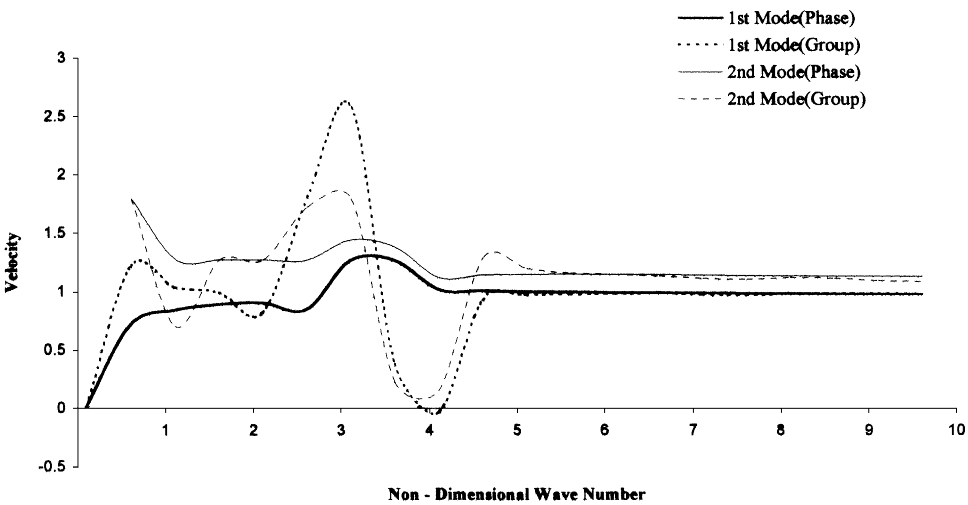

Fig. 2. Variation of phase and group velocities of flexural modes with wave number.

continuously repeated for different values of the non-dimensional wave number kh to obtain the phase velocity. Here the value of velocity has been allowed to iterate approximately for 50 iterations to make it converge in order to achieve the accuracy up to four decimal places. We have used terms of order $(kh)^4$ of the series and calibrated the data for the range $0 \leqslant kh \leqslant 10$ of wave number, without affecting the accuracy level here. The group velocity is also obtained by adopting the same procedure.

The graphs of phase and group velocities of extensional modes given in Fig. 1 look similar to the corresponding graphs of Tolstoy and Usdin [23,24] except that these are slightly modified due to the anisotropic effects of the material. The graphs of these quantities for first (fundamental) mode have the common limit, viz., velocity of longitudinal thin plate wave at low frequencies (long waves) and tend to the velocity of Rayleigh surface waves for short waves ($kh \to \infty$). The velocity of the second mode is approximated reasonably well for short waves and intermediate wavelength and it may be associated with same antisymmetric motion of the plate as in Refs. [19,25,26]. The group velocities are found to be in agreement and no negative group velocity for extensional modes is noticed in this case. In contrast to Ref. [5] the velocity spectrum behavior of the second mode is found to be in agreement for long-wave approximation as the theory is substantiated best ($kh \to 0$) and numerical results at short waves ($kh \to \infty$) have been affected by a greater approximation error. This acceptable distinction to Losin [5] happens because of his mistake in calculations of some of the coefficients in the dispersion relations.

From Fig. 2 it is observed that the phase velocity of the fundamental (first) mode of flexural mode increases from the origin for long waves and tends to the common limit, viz., velocity of Rayleigh surface wave for short waves as predicted by the exact plate theory and are found to be perfectly approximated by the curves in Fig. 2 and are in agreement with the corresponding curves of Ref. [1,21,23–26] except for the modification due to anisotropic effects of the material. The phase velocity of the second mode decreases continuously from infinity for long waves and approaches the velocity of shear mode for short waves. The correspondence between the group velocities is also found to be good with the exception that it is found to be slightly more than the velocity of shear mode at high frequencies in the case of first mode and increases to attain its maxima at $kh = 3$ and then decreases to its minimum (negative) value in the neighbourhood of $kh = 4$. It then increases to become close to the velocity of Rayleigh surface wave for short waves. The group velocity of second mode is found to be comparably in agreement. The appearance of the negative group velocity in a small neighbourhood of $kh = 4$ in Fig. 2 is also noticed. Such an effect was detected by Tolstoy and Usdin [24], and mentioned by some authors, e.g., Mindlin [26], Redwood [27] and Losin [4].

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